

NOTE ON RESTRICTION MAPS OF CHOW RINGS TO WEYL GROUP INVARIANTS

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ABSTRACT. Let G be an algebraic group over \mathbb{C} corresponding a compact simply connected Lie group. When $H^*(G)$ has p -torsion, we see $\rho_{CH}^* : CH^*(BG) \rightarrow CH^*(BT)^{W_G(T)}$ is always not surjective. We also study the algebraic cobordism version ρ_Ω^* . In particular when $G = Spin(7)$ and $p = 2$, we see each Griffiths element in $CH^*(BG)$ is detected by an element in $\Omega^*(BT)$,

1. INTRODUCTION

Let p be a prime number. Let G be a compact Lie group and T the maximal torus. Let us write $H^*(-) = H^*(-; \mathbb{Z}_{(p)})$, and BG, BT classifying spaces of G, T . Let $W = W_G(T) = N_G(T)/T$ be the Weyl group and $Tor \subset H^*(BG)$ be the ideal generated by torsion elements. Then we have the restriction map

$$\rho_H^* : H^*(BG) \rightarrow H^*(BG)/Tor \subset H^*(BT)^W$$

by using the Becker-Gottlieb transfer.

It is well known that when $H^*(G)$ is p -torsion free (hence $H^*(BG)$ is p -torsion free), then ρ_H^* is surjective. However when $H^*(G)$ has p -torsion, there are cases that ρ_H^* are not surjective, which are founded by Feshbach [Fe].

Let us write by $G_{\mathbb{C}}, T_{\mathbb{C}}$ the reductive group over \mathbb{C} and its maximal torus corresponding the Lie group G, T . Let us write simply $CH^*(BG) = CH^*(BG_{\mathbb{C}})_{(p)}$, $CH^*(BT) = CH^*(BT_{\mathbb{C}})_{(p)}$ the Chow rings of $BG_{\mathbb{C}}$ and $BT_{\mathbb{C}}$ localized at p . We consider the Chow ring version

$$\rho_{CH}^* : CH^*(BG) \rightarrow CH^*(BG)/Tor \subset CH^*(BT)^W \cong H^*(BT)^W.$$

Our first observation is

Theorem 1.1. *Let G be simply connected. If $H^*(G)$ has p -torsion, then the map ρ_{CH}^* is always not surjective.*

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In the proof, we use an element $x \in H^4(BG)$ with $\rho_H(x) \notin \text{Im}(\rho_{CH}^*)$. Hence $x \notin \text{Im}(cl)$ for the cycle map $cl : CH^*(BG) \rightarrow H^*(BG)$. The corresponding element $1 \otimes x \in CH^*(B\mathbb{G}_m \times BG)$ is the element founded as a counterexample for the integral Hodge and hence the integral Tate conjecture in [Pi-Ya].

Next, we consider elements in *Tor*. To study torsion elements, we consider the following restriction map

$$\text{res}_H : H^*(BG) \rightarrow \Pi_{A:\text{abelian} \subset G} H^*(BA)^{W_G(A)}.$$

There are cases such that res_H is not injective, while for many cases res_H are injective. We consider the Chow ring version ([To1,2]) of the above restriction map

$$\text{res}_{CH} : CH^*(BG) \rightarrow \Pi_{A:ab.} CH^*(BA)^{W_G(A)} \subset \Pi_{A:ab.} H^*(BA)^{W_G(A)}.$$

In general res_{CH} has non zero kernel. In particular, elements in $\text{Ker}(cl)$ (i.e. Griffiths elements) for the cycle map $cl : CH^*(BG) \rightarrow H^*(BG)$ are always in $\text{Ker}(\text{res}_{CH})$. Namely Griffiths elements are not detected by res_{CH} .

On the other hand, if the Totaro conjecture

$$CH^*(BG) \cong BP^*(BG) \otimes_{BP^*} \mathbb{Z}_{(p)}$$

(for the Brown-Peterson cohomology $BP^*(-)$) is correct, then of course all elements in $CH^*(BG)$ are detected by elements in $BP^*(BG)$. We show that there is a case that Griffiths elements are detected by ρ_Ω^* the restriction for algebraic cobordism theory $\Omega^*(-)$.

Let $\Omega^*(X) = MGL^{2*,*}(X) \otimes_{MU_{(p)}^*} BP^*$ be the BP -version of the algebraic cobordism defined by Voevodsky, Levine-Morel ([Vo1], [Le-Mo1,2]) such that $CH^*(X) \cong \Omega^*(X) \otimes_{BP^*} \mathbb{Z}_{(p)}$. In particular, we consider the case $G = Spin(7)$ and $p = 2$. We note that there are (non zero) Griffiths elements in $CH^*(BG)$.

Theorem 1.2. *Let $G = Spin(7)$ and $p = 2$. Then each Griffiths element (in $CH^*(BG)$) is detected by an element in $\Omega^*(BT)^W \cong BP^*(BT)^W$.*

In §2 we study the map ρ_H^* for the ordinary cohomology theory, and recall Feshbach's result. In §3, we study the Chow ring version and show Theorem 1.1. In §4, we study the case $G = Spin(n)$. In §5, we study the BP^* -version and the algebraic cobordism version for the restriction ρ^* . In §6, we write down the case $G = Spin(7)$ quite explicitly, and show Theorem 1.2. In the last section, we note some partial results for the exceptional group $G = F_4$ and $p = 3$.

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2. COHOMOLOGY THEORY AND FESHBACH THEOREM

Let p be a prime number. Let G be a compact Lie group and T the maximal Torus. Then we have the restriction map

$$\rho_H^* : H^*(BG) \rightarrow H^*(BT)^W$$

where $H^*(-) = H^*(-; \mathbb{Z}_{(p)})$, BG, BT are classifying spaces and $W = W_G(T) = N_G(T)/T$ is the Weyl group.

It is well known that when $H^*(G)$ is p -torsion free, then ρ_H^* is surjective (and hence is isomorphic). However when $H^*(G)$ has p -torsion, there are cases that ρ_H^* are not surjective by Feshbach.

For a connected compact Lie group G , we have the Becker-Gottlieb transfer $\tau : H^*(BT) \rightarrow H^*(BG)$ such that $\tau\rho_H^* = \chi(G/T)$ for the Euler number $\chi(-)$, and $\rho_H^*\tau(x) = \chi(G/T)x$ for $x \in H^*(BT)^W$. Let $\chi(G/T) = N$ and Tor be the ideal of $H^*(BG)$ generated by torsion elements. Then we have the injections

$$N \cdot H^*(BT)^W \subset H^*(BG)/Tor \subset H^*(BT)^W.$$

Feshbach found good criterion to see ρ_H^* is surjective.

Theorem 2.1. (Feshbach [Fe]) *The restriction ρ_H^* is surjective if and only if $(H^*(BG)/Tor) \otimes \mathbb{Z}/p$ has not nonzero nilpotent elements.*

Proof. (Feshbach) First note that $H^*(BT) \cong \mathbb{Z}_{(p)}[t_1, \dots, t_\ell]$ for $|t_i| = 2$. Hence if $x^m = px'$ in $H^*(BT)$, then $x = px''$ for $x'' \in H^*(BT)$. Moreover if $x \in H^*(BT)^W$, then so is x' since $H^*(BT)$ is p -torsion free. Thus we see $H^*(BT)^W \otimes \mathbb{Z}/p$ has no non zero nilpotent elements.

Assume that ρ_H^* is not surjective, and $x \in H^*(BT)^W$ but $x \notin Im(\rho_H^*)$. Let $s \geq 1$ be the smallest number such that $p^s x = \rho_H^*(y)$ for some $y \in H^*(BG)$. Hence $y \neq 0 \mod(p)$. Then

$$\rho_H^*(y^N) = (p^s x)^N = p^{sN} x^N \in pN \cdot H^*(BT)^W \subset pIm(\rho_H^*).$$

This means y is a nonzero nilpotent element in $(H^*(BG)/Tor) \otimes \mathbb{Z}/p$. \square

Using this theorem, Feshbach [Fe] showed ρ_H^* is surjective for $G = G_2, Spin(n)$ for $n \leq 10$, and is not surjective for $Spin(11), Spin(12)$. Wood [Wo] showed that $Spin(13)$ is not surjective but $Spin(n)$ for $14 \leq n \leq 18$ are surjective. Benson and Wood solved this problem completely, namely ρ^* is not surjective if and only if $n \geq 11$ and $n = 3, 4, 5 \mod(8)$.

For odd prime, we consider $\text{mod}(p)$ version

$$\rho_{H/p} : H^*(BG; \mathbb{Z}/p) \rightarrow H^*(BT; \mathbb{Z}/p)^W \cong (H^*(BT)/p)^W.$$

It is known that $\rho_{H/p}^*$ are surjective when $G = F_4$ for $p = 3$ by Toda [Tod] using a completely different arguments. Also different arguments (but without computations of $H^*(BT)^W$ for concrete cases), Kameko and Mimura [Ka-Mi] prove that $\rho_{H/p}^*$ are surjective when $G = E_6, E_7$ for $p = 3$ and $G = E_8$ for $p = 5$. (The only remain case is $G = E_8$, $p = 3$ for odd primes.)

Kameko-Mimura get more strong result. Recall the Milnor Q_i operation

$$Q_i : H^*(X; \mathbb{Z}/p) \rightarrow H^{*+2p^i-1}(X; \mathbb{Z}/p)$$

defined by $Q_0 = \beta$ and $Q_{i+1} = [P^{p^i} Q_i, Q_i P^{p^i}]$ for the Bockstein β and the reduced powers P^j .

Theorem 2.2. (*Kameko-Mimura [Ka-Mi]*) *Let $G = F_4, E_6, E_7$ for $p = 3$ or E_8 for $p = 5$. Let us write a generator by x_4 in $H^4(BG) \cong \mathbb{Z}_{(p)}$. Then we have*

$$H^*(BT; \mathbb{Z}/p)^W \cong H^{\text{even}}(BG; \mathbb{Z}/p)/(Q_1 Q_2 x_4).$$

Corollary 2.3. *For (G, p) in the above theorem, ρ_H^* is surjective.*

Since its Q_0 -image is zero, we can identify the element $Q_1 Q_2(x_4)$ is in $H^*(BG)$ and p -torsion. The above corollary is immediate from the following lemma.

Lemma 2.4. *If the composition*

$$(H^*(BG)/\text{Tor}) \otimes \mathbb{Z}/p \rightarrow H^*(BT)^W/p \rightarrow H^*(BT; \mathbb{Z}/p)^W$$

is injective, then ρ_H^ is surjective.*

Proof. Let ρ_H^* be not surjective and $y \in H^*(BT)^W$ with $y \notin \text{Im}(\rho_H^*)$. Then $p^s y = \rho_H^*(x)$ for $s \geq 1$ and an additive generator $x \in H^*(BG)/\text{Tor}$. Of course $x = 0 \in (H^*(BT)/p)^W$. \square

3. CHOW RINGS

Let us write by $G_{\mathbb{C}}, T_{\mathbb{C}}$ the reductive group over \mathbb{C} and its maximal torus corresponding the Lie group G . Let $CH^*(BG) = CH^*(BG_{\mathbb{C}})_{(p)}$ be the Chow ring of $BG_{\mathbb{C}}$ localized at p .

The arguments of Feshbach also work for Chow rings since the Becker-Gottlieb transfer is constructed by Totaro [To2].

Lemma 3.1. *The restriction map ρ_{CH}^* of Chow rings is surjective if and only if $(CH^*(BG)/T) \otimes \mathbb{Z}/p$ has not nonzero nilpotent elements.*

However if $H^*(G)$ has p -torsion and G is simply connected, then $(CH^*(BG)/Tor) \otimes \mathbb{Z}/p$ always has non zero nilpotent elements. In fact, $c_2 = px_4 \in CH^4(BG)$ in the proof of Theorem 3.3 below, is nilpotent in $(CH^*(BG)/(Tor)) \otimes \mathbb{Z}/p$. However from the proof of the above lemma, we note

Corollary 3.2. *If $x \in CH^*(BT)^W$ but $x \notin Im(\rho_{CH}^*)$, then there is $y \in CH^*(BG)$ such that $\rho_{CH}^*(y) = p^s x$ for some $s \geq 1$ and y is non zero nilpotent element in $(CH^*(BG)/(Tor)) \otimes \mathbb{Z}/p$.*

Voevodsky [Vo1,2] defined the Milnor operation Q_i on the mod p motivic cohomology (over a perfect field k of any $ch(k)$)

$$Q_i : H^{*,*'}(X; \mathbb{Z}/p) \rightarrow H^{*+2(p^n-1), *'+p^n-1}(X; \mathbb{Z}/p)$$

which is compatible with the usual topological Q_i by the realization map $t_{\mathbb{C}} : H^{*,*'}(X; \mathbb{Z}/p) \rightarrow H^*(X(\mathbb{C}); \mathbb{Z}/p)$ when $ch(k) = 0$. In particular, note for smooth X ,

$$Q_i|CH^*(X)/p = Q_i|H^{2*,*}(X; \mathbb{Z}/p) = 0.$$

(See §2 in [Pi-Ya] for details.) We will prove the following theorem without using Feshbach theorem (Lemma 3.1).

Theorem 3.3. *Let G be simply connected and $H^*(G)$ has p -torsion. Then the restriction map*

$$\rho_{CH}^* : CH^2(BG) \rightarrow CH^2(BT)^W$$

is not surjective.

Proof. (See §2, 3 in [Pi-Ya].) At first, we note that $H^*(BT)^W \cong CH^*(BT)^W$ since $H^*(BT) \cong CH^*(BT)$. If $H^*(G)$ has p -torsion, then G has a subgroup isomorphic to G_2 (resp. F_4 , E_8) for $p = 2$ (resp. $p = 3, 5$). (For details, see [Ya2] or §3 in [Pi-Ya].) We prove the theorem for $p = 2$ but the other cases are proved similarly.

It is known that the inclusion $G_2 \subset G$ induces a surjection $H^4(BG) \rightarrow H^4(BG_2) \cong \mathbb{Z}_{(2)}$ and let us write by x_4 its generator. Then it is also known $Q_1 x_4 \neq 0$ in $H^*(BG_2; \mathbb{Z}/2)$ where Q_1 is the Milnor operation. Therefore $x_4 \in H^4(BG_2)$ is not in the image of the cycle map

$$cl : CH^2(BG_2) \rightarrow H^4(BG_2).$$

On the other hand, the element $2x_4$ is in $Im(cl)$ because it is represented by the second Chern class c_2 . Since $\rho^* \otimes \mathbb{Q}$ is an isomorphism, $\rho_H^*(x_4) \neq 0$. But $\rho_H^*(x_4)$ is not in the image ρ_{CH}^* . \square

Remark. The condition of simply connected is necessary. By Vistoli ([Vi], [Ka-Mi]), it is known that ρ_{CH}^* is surjective for $G = PGL(p)$.

Remark. The above theorem is also proved by seeing that x_4 is not generated by Chern classes, since $CH^2(X)$ is always generated by Chern classes [To2].

Recall that for a smooth projective complex variety X , the integral Hodge conjecture is that the cycle map

$$cl_{/Tor} : CH^*(X) \rightarrow H^{2*}(X)/Tor \cap H^{*,*}(X)$$

is surjective where $H^{*,*}(X) \subset H^{2*}(X; \mathbb{C})$ is the submodule generated by $(*, *)$ -forms. Since $px_4 = c_2$ in the proof of the above theorem and $c_2 \in H^{*,*}(X)$, we see $x_4 \in H^{*,*}(X)$.

We know [To1], [Pi-Ya] that $B\mathbb{G}_m \times BG$ can be approximated by smooth projective varieties. Hence counterexamples for the integral Hodge conjecture with $X = B\mathbb{G}_m \times BG$ give the examples such that ρ_{CH}^* is not surjective.

Lemma 3.4. *Let $1 \otimes y \notin Im(cl_{/Tor}) \subset H^*(B\mathbb{G}_m \times BG)$ be a counterexample of the integral Hodge conjecture. Then it gives an example such that ρ_{CH}^* is not surjective, namely, $\rho_H^*(y) \notin Im(\rho_{CH}^*)$.*

Proof. First note that $\rho_{H/Tor}^* : H^*(BG)/Tor \rightarrow H^*(BT)^W$ is injective. Since $CH^*(BT)^W \cong H^*(BT)^W$, we note $\rho_{CH}^* = \rho_{H/Tor}^* cl_{/Tor}$. Therefore $y \notin Im(cl_{/Tor})$ implies that $\rho_H^*(y) \notin Im(\rho_{H/Tor}^* cl_{/Tor}) = Im(\rho_{CH}^*)$. \square

For each prime p , there are counterexamples $X = B\mathbb{G}_m \times BG$ for the integral Hodge conjecture, while they are not simply connected. Indeed, Kameko, Antineau and Tripaphy ([Ka1,2], [An], [Tr]) show this for $G = (SL_p \times SL_p)/\mathbb{Z}/p$ and $SU(p^2)/\mathbb{Z}/p$. Hence they give the examples such that ρ_{CH}^* are not surjective for non simply connected and all p cases. They proved these facts by using Chern classes.

We also note its converse. Recall [Pi-Ya] that the integral Tate conjecture over a finite field k is the $ch(k) > 0$ version of the integral Hodge conjecture.

Lemma 3.5. *Let $x \in H^*(BT)^W$ such that $x \notin \rho_{CH}^*$ but $x = \rho_H^*(y)$. Moreover let $p^s y$ be represented by Chern class for $s \geq 1$. Then $1 \otimes y \in H^*(B\mathbb{G}_m \times BG)$ gives a counterexample of the integral Hodge conjecture. It also gives a counterexample of the integral Tate conjecture for a finite field k of $ch(k) \neq p$*

Proof. Since $p^s y$ is represented by a Chern class, we see $p^s y \in Im(cl)$. Hence it is contained in $H^{*,*}(B\mathbb{G}_m \times BG)$. Hence so is y . Since $x \notin \rho_{CH}^*$, we see $y \notin cl_{/Tor}$. For arguments for the integral Tate conjecture see [Pi-Ya]. \square

4. $Spin(n)$ FOR $p = 2$

In this section, we study Chow rings for the cases $G = Spin(n)$, $p = 2$. Recall that the $mod(2)$ cohomology is given by Quillen [Qu1]

$$H^*(BSpin(n); \mathbb{Z}/2) \cong \mathbb{Z}/2[w_2, \dots, w_n]/J \otimes \mathbb{Z}/2[e]$$

where $e = w_{2h}(\Delta)$ and $J = (w_2, Q_0 w_2, \dots, Q_{h-2} w_2)$. Here w_i is the Stiefel-Whitney class for the natural covering $Spin(n) \rightarrow SO(n)$. The number 2^h is the Radon-Hurewitz number, dimension of the spin representation Δ (which is the representation $\Delta|_C \neq 0$ for the center $C \cong \mathbb{Z}/2 \subset Spin(n)$). The element e is the Stiefel-Whitney class w_{2h} of the spin representation Δ .

Hereafter this section we always assume $G = Spin(n)$ and $p = 2$.

By Kono [Ko], it is known that $H^*(BG; \mathbb{Z})$ has no higher 2-torsion, that is

$$H(H^*(BG; \mathbb{Z}/2); Q_0) \cong (H^*(BG)/Tor) \otimes \mathbb{Z}/2$$

where $H(A; Q_0)$ is the homology of A with the differential $d = Q_0$.

For ease of arguments, let n be odd i.e., $n = 2k + 1$. Let T' be a maximal Torus of $SO(n)$ and $W' = W_{SO(n)}(T')$ its Weyl group. Then $W' \cong S_k^\pm$ is generated by permutations and change of signs so that $|S_k^\pm| = 2^k k!$. Hence we have

$$H^*(BT')^{W'} \cong \mathbb{Z}_{(2)}[p_1, \dots, p_k] \subset H^*(BT') \cong \mathbb{Z}_{(2)}[t_1, \dots, t_k], \quad |t_i| = 2$$

where the Pontriyagin class p_i is defined by $\Pi_i(1 + t_i^2) = \sum_i p_i$.

For the projection $\pi : Spin(n) \rightarrow SO(n)$, the maximal torus of $Spin(n)$ is given $\pi^{-1}(T')$ and $W = W_{Spin(n)}(T) \cong W'$. To seek the invariant $H^*(BT)^W$ is not so easy, since the action $W \cong S_k^\pm$ is not given by permutations and change of signs. Benson and Wood decided the $H^*(BT')^{W'}$ -algebra structure of $H^*(BT)^W$ (Theorem 7.1 in [Be-Wo]) and proved

Theorem 4.1. (*Benson-Wood*) *Let $G = Spin(n)$ and $p = 2$. Then ρ_H^* is not surjective if and only if $n \geq 11$ and $n = 3, 4, 5 \pmod{8}$ (i.e., the quaternion case).*

Hereafter to study the Chow ring version, we assume $Spin(n)$ is in real case [Qu1], that is $n = 8\ell - 1, 8\ell, 8\ell + 1$ (hence ρ^* is surjective and $h = 4\ell - 1, 4\ell - 1, 4\ell$ respectively).

In this case, it is known [Qu1] that each maximal elementary abelian 2-group A has $rank_2 A = h + 1$ and

$$e|A = \Pi_{x \in H^1(B\bar{A}; \mathbb{Z}/2)}(z + x)$$

where we identify $A \cong C \oplus \bar{A}$ and $H^1(B\bar{A}; \mathbb{Z}/2) \cong \mathbb{Z}/2\{x_1, \dots, x_h\}$ is the $\mathbb{Z}/2$ vector space generated by x_1, \dots, x_h , and

$$H^*(BC; \mathbb{Z}/2) \cong \mathbb{Z}/2[z], \quad H^*(B\bar{A}; \mathbb{Z}/2) \cong \mathbb{Z}/2[x_1, \dots, x_h].$$

The Dickson algebra is written as a polynomial algebra

$$\mathbb{Z}/2[x_1, \dots, x_h]^{GL_h(\mathbb{Z}/2)} \cong \mathbb{Z}/2[d_0, \dots, d_{h-1}].$$

where d_i is defined as

$$e|A = z^{2^h} + d_{h-1}z^{2^{h-1}} + \dots + d_0z.$$

Hence we can [Qu1] also identify $d_i = w_{2^h-2^i}(\Delta) \in H^*(BSpin(n); \mathbb{Z}/2)$.

Lemma 4.2. (*Lemma 2.1 in [Sc-Ya]*) *Milnor operations act on d_i by*

$$\begin{aligned} Q_{h-1}d_i &= d_0d_i, & Q_{j-1}d_j &= d_0, \text{ for } 1 \leq j, \\ Q_id_j &= 0 \text{ for } i < n-1 \text{ and } i \neq j-1. \end{aligned}$$

Lemma 4.3. (*Corollary 2.1 in [Sc-Ya]*) *We have*

$$Q_{h-1}e = d_0e \quad \text{and} \quad Q_ke = 0 \quad \text{for } 0 \leq k \leq h-2.$$

Theorem 4.4. *Let $T \subset G = Spin(n)$ for $n = 8\ell, 8\ell \pm 1$. There is an $e' \in CH^*(BT)^W$ such that $e' \notin Im(\rho_{CH}^*)$ and $\rho_H^*(e) = e' \bmod(2)$.*

Proof. First note that $e|C = z^{2^h}$, which is not in the Q_0 -image, and hence e itself is not. From the preceding Lemma 4.2 we see $Q_0e = 0$. By Kono's result, we see

$$0 \neq e \in H(H^*(BG; \mathbb{Z}/2); Q_0) \cong (H^*(BG)/Tor) \otimes \mathbb{Z}/2.$$

Take $e'' \in H^*(BG)/Tor$ with that $e'' = e \bmod(2)$. Then

$$e' = \rho_H^*(e'') \neq 0 \text{ in } H^*(BG)/Tor \subset H^*(BT)^W.$$

From the preceding Lemma 4.2, $Q_{h-1}(e) \neq 0$. hence we see $e' \notin \rho_{CH}^*$ by the existence of Q_i in the motivic cohomology by Voevodsky. \square

Let $\Delta_{\mathbb{C}}$ be the complex representation induced from the real representation Δ . Then we see (see Theorem 4.2 in [Sc-Ya])

$$c_{2^{h-1}}(\Delta_{\mathbb{C}})|C = 2w_{2^h}|C = 2z^{2^h}.$$

Of course this element $c_{2^{h-1}}(\Delta_{\mathbb{C}})$ is in the Chow ring $CH^*(BG)$. hence we see that we can take $2e' \in Im(\rho_{CH}^*)$.

From the result by Benson-Wood, we know ρ_H^* is surjective in this (real) case. Hence from Lemma 3.5 (or $Q_{h-1}(e) \neq 0$), we have

Corollary 4.5. *Let $X = B\mathbb{G}_m \times BSpin(n)$ with $n = 8\ell, 8\ell \pm 1$. The element $1 \otimes e \in H^{2^h}(X) \cap H^{2^{h-1}, 2^{h-1}}(X)$ gives a counterexample for the integral Hodge and the integral Tate conjectures, namely $1 \otimes e \notin Im(cl_{H/Tor})$.*

5. COBORDISM

Let $BP^*(X)$ be the Brown-Peterson cohomology theory with the coefficients ring $BP^* = \mathbb{Z}_{(p)}[v_1, v_2, \dots]$ of degree $|v_i| = -2(p^i - 1)$. Let $\Omega^*(X) = MGL^{2*,*}(X) \otimes_{MU^*} BP^*$ be the BP^* -version of the algebraic cobordism ([Vo1], [Le-Mo1,2]) such that $\Omega^*(X) \otimes_{BP^*} \mathbb{Z}_{(p)} \cong CH^*(X)$.

We consider the cobordism version of the map ρ_H^*

$$\rho_\Omega^* : \Omega^*(BG) \rightarrow \Omega^*(BT)^W \cong BP^*(BT)^W.$$

Although \mathbb{A}^1 -homotopy category has the Becker-Gottlieb transfer (this fact is announced in [Ca-Jo]), we see

$$\tau \cdot \rho_\Omega^* = \chi(G/T) \bmod(v_1, v_2, \dots)$$

which is not $\chi(G/T)$ in general. So we can not have the Ω^* -version of Feshbach's theorem.

We are interesting in an element $x \in \Omega^*(BT)^W$ which is in $Im(\rho_\Omega^*)$. Of course, it is torsion free in $\Omega^*(X)$, but there is a case such that

$$0 \neq x \in CH^*(BG)/p \cong \Omega^*(BG) \otimes_{BP^*} \mathbb{Z}/p$$

and x is p -torsion in $CH^*(BG)$.

Lemma 5.1. *Let $f \in H^*(BT)^W$, $f \neq 0 \bmod(p)$, and identify $f \in gr\Omega^*(BT) \cong \Omega^* \otimes H^*(BT)$. Let $f \notin Im(\rho_\Omega^*)$ but $v_m f \in Im(\rho_\Omega^*)$. Then $v_j f \in Im(\rho_\Omega^*)$ for all $0 \leq j \leq m$. Namely, there is $c_j \in \Omega^*(BG)$ such that $\rho_\Omega^*(c_j) = v_j f$,*

$$c_j \neq 0 \in \Omega^*(BG) \otimes_{BP^*} \mathbb{Z}/p \cong CH^*(BG)/p,$$

moreover $pc_j = 0$ in $CH^*(BG)$ for $j > 0$.

Proof. We consider the Landweber-Novikov cohomology operation r_a in $gr\Omega^*(BT) \cong \Omega^* \otimes H^*(BT)$. By Cartan formula,

$$r_a(v_m f) = \sum_{a=a'+a''} r_{a'}(v_m) r_{a''}(f).$$

Here $r_{a''}(f) = 0$ for $|a''| > 0$ in $gr\Omega^*(BT) \cong \Omega^* \otimes H^*(BT)$. We have operations $r_{\beta_j}(v_m) = v_j$ for $j \leq m$, and we have the first statement.

From the assumption, f itself is not in the cycle map ρ_{Ω^*} . Hence $v_j f$ is a BP^* -module generator in $\Omega^*(BT)^W \cap Im(\Omega^*(BG))$. Hence it is also non zero in $CH^*(BG)/p$. Since $pv_j f = v_j pf \in v_j Im(\Omega^*(BG))$, we have $pc_j = 0 \in CH^*(BG)$. \square

We consider the Atiyah-Hirzebruch spectral sequence (AHss)

$$E_2^{*,*'} \cong H^*(X; BP^{*'}) \implies BP^*(X)$$

It is known that

$$(*) \quad d_{2p^i-1}(x) = v_i \otimes Q_i(x) \mod(p, v_1, \dots, v_{i-1}).$$

In general, there are many other types of non zero differential. However we consider cases that differentials are only of this form.

Lemma 5.2. *Let $X = BSpin(n)$ and $n = 8\ell, 8\ell \pm 1$. In AHss for $BP^*(X)$, assume all non zero differentials are of form $(*)$. Then $pe, v_1e, \dots, v_{h-2}e$ are all permanent cycles.*

Proof. We use Lemma 4.2 in the preceding section. First recall $Q_i(d_0) = 0$, $Q_i(e) = 0$ for $i < h-1$. Therefore d_0e exists in E_{2h-1} .

Since $Q_{j-1}d_j = d_n$ and $Q_k(d_j) = 0$ for $k < j-1$, the differential in AHss is

$$d_{2j-1}(d_je) = v_{j-1} \otimes Q_{j-1}(d_je) = v_{j-1}d_0e.$$

Hence we have $(2, v_1, v_2, \dots, v_{h-2})(d_0e) = 0$ in $E_{2h-1}^{*,*}$.

Now we study the differential

$$d_{2h-1}(e) = v_{h-1}Q_{h-1}(e) = v_{h-1}d_0e.$$

Note that e is BP^* -free in $E_{2h-1}^{*,*}$, since $e|C = z^{2h}$ and $e \notin \text{Im}(Q_i)$. Hence we have

$$\text{Ker}(d_{2h-1}) \cap BP^*\{e\} \cong \text{Ideal}(2, v_1, \dots, v_{h-2})\{e\}.$$

(In this paper, $R\{a, b, \dots\}$ means the R -free module generated by a, b, \dots) By the assumption $(*)$ for differentials, $pe, v_1e, \dots, v_{h-2}e$ are all permanent cycles. \square

For $7 \leq n \leq 9$, AHss converging $BP^*(BSpin(n))$ is computed in [Ko-Ya], ([Sc-Ya] also), and it is known that $(*)$ is satisfied.

Corollary 5.3. *For $n = 7, 8$ (resp. $n = 9$), the elements pe, v_1e (resp. pe, v_1e, v_2e) are in $\text{Im}(\rho_{BP}^*) \subset BP^*(BT)^W$ (but e itself is not).*

Let $K(n)^*(X)$ be the Morava K -theory with the coefficients ring $K(n)^* \cong \mathbb{Z}/p[v_n, v_n^{-1}]$, and $AK(n)^*(X) = AK(n)^{2*,*}(X)$ its algebraic version. Here we consider an assumption such that

$$(**) \quad AK(n)^*(BG) \rightarrow K(n)^*(BG) \quad \text{is surjective.}$$

It is known by Merkurjev (see [To1] for details) that $AK^*(BG) \cong K^*(BG)$ for the algebraic K -theory $AK^*(X)$ and the complex K -theory $K^*(X)$, which induces $AK(1)^*(BG) \cong K(1)^*(BG)$. Hence $(**)$ is correct when $n = 1$ for all G .

Lemma 5.4. *Let $X = BSpin(n)$, $n = 8\ell, 8\ell \pm 1$ and suppose $(*), (**)$. Moreover let $h \geq 3$. Then $v_{h-2}e \in \text{Im}(\rho_\Omega^*)$, and hence there is $c_i \in CH^*(X)$ for $0 \leq i \leq h-2$ in Lemma 5.1.*

Proof. First note $0 \neq v_{h-2}e \in K(h-2)^*(X)$ (hence so is e). On the other hand

$$AK(h-2)^*(X) \cong K(h-2)^* \otimes CH^*(X)/I$$

for some ideal I of $CH^*(X)$. Therefore there is an element $c \in CH^*(X)$ which corresponds $v_{h-2}^s e$ that is $cl_\Omega(c) = v_{h-2}^s e$ for $cl_\Omega : \Omega^*(X) \rightarrow BP^*(X)$. Since $e \notin Im(cl_\Omega)$, we see s must be positive. The possibility of

$$|v_{h-2}^s e| = -2(2^{h-2} - 1)s + 2^h > 0$$

is $s = 1$ or $s = 2$. Note $|v_{h-2}^2 e| = 4$. However it is known by Totaro (Theorem 15.1 in [To2]),

$$cl : CH^2(X) \rightarrow H^4(X) \text{ is injective.}$$

Hence $s = 1$ and $cl_\Omega(c) = v_{h-1}e$. \square

Corollary 5.5. *For $X = BSpin(7)$, there is an element $c \in CH^3(X)$ such that $c \neq 0 \in CH^*(X)/2$, $cl(c) = 0$ and $\rho_\Omega^*(c) \neq 0 \in \Omega^*(BT)^W$.*

6. $Spin(7)$ FOR $p = 2$

Let G be a compact Lie group. Consider the restriction map

$$res_{H/p} : H^*(BG; \mathbb{Z}/p) \rightarrow Lim_{V:el.ab} H^*(BV; \mathbb{Z}/p)^{W_G(A)}$$

where $W_G(A) = N_G(A)/C_G(A)$ and V ranges in the conjugacy classes of maximal elementary abelian p -groups. Quillen [Qu2] showed this $res_{H/p}$ is an F -isomorphism (i.e. its kernel and cokernel are generated by nilpotent elements). We consider its integral version

$$res_H : H^*(BG) \rightarrow \Pi_{A:ab} H^*(BA)^{W_G(A)},$$

where A ranges in the conjugacy classes of maximal abelian subgroups of G .

Hereafter this section, we assume $G = Spin(7)$ and $p = 2$ and hence $h = 3$. The number of conjugacy classes of the maximal abelian subgroups of G is two, one is the torus T and the other is $A' \cong (\mathbb{Z}/2)^4$ which is not contained in T . The Weyl group is $W_G(A') \cong \langle U, GL_3(\mathbb{Z}/2) \rangle \subset GL_4(\mathbb{Z}/2)$ where U is the maximal unipotent group in $GL_4(\mathbb{Z}/2)$. It is well known

$$\begin{aligned} H^*(BG; \mathbb{Z}/2) &\cong H^*(BA'; \mathbb{Z}/2)^{W_G(A')} \\ &\cong \mathbb{Z}/2[w_4, w_6, w_7, w_8], \quad Q_0 w_6 = w_7 \end{aligned}$$

where w_i for $i \leq 7$ (resp. $i = 8$) are the Stiefel-Whitney class for the representation induced from $Spin(7) \rightarrow SO(7)$ (resp. the spin representation Δ and hence $w_8(\Delta) = e$).

Since $H^*(BG)$ has just 2-torsion by Kono, and hence the restriction map res_H injects $Tor \subset H^*(BG)$ into $H^*(BA'; \mathbb{Z}/2)^{W_G(A')}$.

Next we consider the integral case. Also note $H^*(BG)$ has just 2-torsion and

$$(H^*(BG)/Tor) \otimes \mathbb{Z}/2 \cong H(H^*(BG; \mathbb{Z}/2); Q_0).$$

Since $Q_0 w_i = 0$ for $i \neq 6$ and $Q_0 w_6 = w_7$, we have

$$H(H^*(BG; \mathbb{Z}/2); Q_0) \cong \mathbb{Z}/2[w_4, c_6, w_8] \quad c_6 = w_6^2.$$

Of course the right hand side ring has no nonzero nilpotent elements. Hence we see that ρ_H^* is surjective and

$$H^*(BT)^W \otimes \mathbb{Z}/2 \cong \mathbb{Z}/2[w_4, c_6, w_8].$$

The integral cohomogy is written as

$$H^*(BG) \cong \mathbb{Z}_{(2)}[w_4, c_6, w_8] \otimes (\mathbb{Z}_{(2)}\{1\} \oplus \mathbb{Z}/2[w_7]\{w_7\}).$$

In particular, we note res_H is injective.

Next we consider the Atiyah-Hirzebruch spectral sequence

$$E_2^{*,*'} \cong H^*(BG) \otimes BP^* \implies BP^*(BG).$$

Its differentials have forms of $(*)$ in §5. Using $Q_1(w_4) = x_7$, $Q_1(e_7) = c_7$, $Q_2(w_8) = w_7 w_8$ and $Q_3(w_7 w_8) = c_7 c_8$, we can compute the spectral sequence (while it is some what complicated)

$$\begin{aligned} gr BP^*(BG) &\cong BP^*[c_4, c_6, c_8]\{1, 2w_4, 2w_8, 2w_4 w_8, v_1 w_8\} \\ &\quad \oplus BP^*/(2, v_1, v_2)[c_4, c_6, c_7, c_8]\{c_7\}/(v_3 c_7 c_8). \end{aligned}$$

Hence $BP^*(BG) \otimes_{BP^*} \mathbb{Z}_{(2)}$ is isomorphic to

$$\begin{aligned} &\mathbb{Z}_{(2)}^*[c_4, c_6, c_8]\{1, 2w_4, 2w_8, 2w_4 w_8, v_1 w_8\}/(2v_1 w_8) \\ &\quad \oplus \mathbb{Z}/2[c_4, c_6, c_7, c_8]\{c_7\}. \end{aligned}$$

On the other hand, the Chow ring of BG is given by Guillot [Gu, Ya]

$$CH^*(BG) \cong BP^*(BG) \otimes_{BP^*} \mathbb{Z}_{(2)}$$

$$\cong \mathbb{Z}_{(2)}[c_4, c_6, c_8] \otimes (\mathbb{Z}_{(2)}\{1, c'_2, c'_4 \cdot c'_6\} \oplus \mathbb{Z}/2\{\xi_3\} \oplus \mathbb{Z}/2[c_7]\{c_7\})$$

where $cl(c_i) = w_i^2$, $cl(c'_2) = 2w_4$, $cl(c'_4) = 2w_8$, $cl(c'_6) = 2w_4 w_8$, and $cl(\xi_3) = 0$, $|\xi_3| = 6$. Note $cl_\Omega(\xi_3) = v_1 w_8$ in $BP^*(BT)^W$, and $\xi_3 = c$ in Corollary 5.5. Hence we have

$$\begin{aligned} CH^*(BG)/Tor &\cong \mathbb{Z}_{(2)}[c_4, c_6, c_8]\{1, c'_2, c'_4 \cdot c'_6\} \\ &\subset \mathbb{Z}_{(2)}[w_4, c_6, w_8] \cong CH^*(BT)^W. \end{aligned}$$

In fact the nilpotent ideal in $(CH^*(BG)/(Tor)) \otimes \mathbb{Z}/2$ is generated by c'_2, c'_4, c'_6 .

Next we consider the Chow rings version for the restriction map

$$res_{CH} : CH^*(BG) \rightarrow \Pi_{A:ab}.CH^*(BA)^{W_G(A)}.$$

Recall $CH^*(BA') \cong \mathbb{Z}_{(2)}[y_1, \dots, y_4]$ with $cl(y_i) = x_i^2$. Hence we have

$$(CH^*(BA')/2)^{W_G(A')} \cong \mathbb{Z}/2[c_4, c_6, c_7, c_8].$$

Since Tor is just 2-torsion, we have

Lemma 6.1. *For the torsion ideal Tor in $CH^*(BG)$*

$$res_{CH}(Tor) \cong \mathbb{Z}/2[c_4, c_6, c_8, c_7]\{c_7\} \subset CH^*(BA').$$

Thus we see that $Ker(res_{CH}) \cong \mathbb{Z}/2[c_4, c_6, c_8]\{\xi_3\}$, which is the ideal of Griffiths elements. We write down the above results.

Theorem 6.2. *Let $(G, p) = (Spin(7), 2)$. Let $Gri f$ be the ideal generated by Griffiths elements and $D = \mathbb{Z}_{(2)}[c_4, c_6, c_8]$. Then we have*

$$\begin{aligned} CH^*(BG)/Tor &\cong D\{1, 2w_4, 2w_8, 2w_4w_8\} \\ &\subset D\{1, w_4, w_8, w_4w_8\} \cong CH^*(BT)^W, \text{ with } w_i^2 = c_i, \\ Tor/Gri f &\cong D/2[c_7]\{c_7\}, \quad Gri f \cong D/2\{\xi_3\}. \end{aligned}$$

There is only one nonzero element ξ_3 in $CH^3(BG)/2$, and this $\xi_3 = c$ in Corollary 5.5 in the preceding section. Thus we see Theorem 1.2 in the introduction.

Corollary 6.3. *Take an element $\xi \in \Omega^*(BG)$ such that $\xi = \xi_3$ in $\Omega^*(BG) \otimes_{BP^*} \mathbb{Z}_{(2)} \cong CH^*(BG)$. We also identify $c_i \in \Omega^*(BG)$. Then*

$$\mathbb{Z}/2[c_4, c_6, c_8]\{\xi\} \subset \Omega^*(BT)^W/2.$$

Theorem 6.4. *Let $G = Spin(7)$ and $p = 2$. We consider a (non natural) map*

$$CH^*(BG) \subset \Omega^*(BG) \rightarrow \Omega^*(BT)$$

(here the first inclusion is just one section of the projection $\Omega^*(BG) \rightarrow CH^*(BG)$). Then the following restriction map is injective

$$res : CH^*(BG) \subset CH^*(BA') \times \Omega^*(BT).$$

Corollary 6.5. *Let $J \subset BP^*$ is the ideal $J = (2^2, 2v_1, v_1^2, v_2, \dots)$ so that $BP^*/J \cong \mathbb{Z}/4\{1\} \oplus \mathbb{Z}/2\{v_1\}$. Let $D = \mathbb{Z}_{(2)}[c_4, c_6, c_8]$. Then*

$$\Omega^*(BG)/J \cong (\Omega^*/J \otimes D\{1, c'_2, c'_4, c'_6, \xi_3\}/(2\xi_3 = v_1c'_4)) \oplus D/2[c_7]\{c_7\}.$$

7. THE EXCEPTIONAL GROUP F_4 , $p = 3$.

In this section, we assume $(G, p) = (F_4, 3)$. (However similar arguments also work for $(G, p) = (E_6, 3), (E_7, 3)$ and $(E_8, 5)$, [Ka-Ya].) Toda computed the $\text{mod}(3)$ cohomology of BF_4 . (For details see [Tod].)

$$H^*(BG; \mathbb{Z}/3) \cong C \otimes D, \quad \text{where}$$

$$\begin{aligned} C &= F\{1, x_{20}, x_{20}^2\} \oplus \mathbb{Z}/3[x_{26}] \otimes \Lambda(x_9) \otimes \{1, x_{20}, x_{21}, x_{26}\} \\ D &= \mathbb{Z}_{(3)}[x_{36}, x_{48}], \quad F = \mathbb{Z}_{(3)}[x_4, x_8]. \end{aligned}$$

Using that $H^*(BG)$ has no higher 3-torsion and $Q_0x_8 = x_9$, $Q_0x_{20} = x_{21}$, $Q_0x_{25} = x_{26}$, we can compute

$$H^*(BG) \cong D \otimes C' \quad \text{where}$$

$$\begin{aligned} C'/\text{Tor} &\cong \mathbb{Z}_{(3)}\{x_4\} \oplus E, \quad \text{where } E = F\{ab|a, b \in \{x_4, x_8, x_{20}\}\} \\ C' \supset \text{Tor} &\cong \mathbb{Z}/3[x_{26}]\{x_{26}, x_{21}, x_9, x_9x_{21}\}. \end{aligned}$$

Note $x_{26} = Q_2Q_1(x_4)$ in Theorem 2.2 and

$$H^*(BT; \mathbb{Z}/p)^W \cong H^{\text{even}}(BG; \mathbb{Z}/3)/(Q_2Q_1x_4) \cong D \otimes F\{1, x_{20}, x_{20}^2\}.$$

Hence we see

$$(H^*(BG)/\text{Tor}) \otimes \mathbb{Z}/3 \cong D/3 \otimes (\mathbb{Z}/3\{1, x_4\} \oplus E) \subset D/3 \otimes F\{1, x_{20}, x_{20}^2\}.$$

Thus from Lemma 2.3, we see ρ_H^* is surjective and

$$H^*(BT)^W \cong H^*(BG)/\text{Tor} \cong D \otimes (\mathbb{Z}_{(3)}\{1, x_4\} \oplus E).$$

Next we consider the Atiyah-Hirzebruch spectral sequence [Ko-Ya]

$$E_2^{*,*'} \cong H^*(BG) \otimes BP^* \implies BP^*(BG).$$

Its differentials have forms of $(*)$ in §5. Using $Q_1(x_4) = x_9$, $Q_1(x_{20}) = x_{25}$, $Q_1(x_{21}) = x_{26}$ and $Q_2x_9 = x_{26}$ we can compute

$$\text{gr}BP^*(BG) \cong D \otimes (BP^* \otimes (\mathbb{Z}_{(3)}\{1, 3x_4\} \oplus E) \oplus BP^*/(3, v_1, v_2)[x_{26}]\{x_{26}\}).$$

Hence we have

$$BP^*(BG) \otimes_{BP^*} \mathbb{Z}_{(3)} \cong D \otimes (\mathbb{Z}_{(3)}\{1, 3x_4\} \oplus E \oplus \mathbb{Z}/3[x_{26}]\{x_{26}\}).$$

Proposition 7.1. *Let $(G, p) = (F_4, 3)$ and $\text{Tor} \supset \text{Grif}$ be the ideal generated by Griffiths elements. Then we have*

$$\begin{aligned} CH^*(BG)/\text{Tor} &\subset D \otimes (\mathbb{Z}_{(3)}\{1, 3x_4\} \oplus E) \subset H^*(BG)/\text{Tor}, \\ \text{Tor}/\text{Grif} &\cong D \otimes \mathbb{Z}/3[x_{26}]\{x_{26}\}. \end{aligned}$$

If Totaro's conjecture is correct, then $\text{Grif} = \{0\}$ and the first inclusion is an isomorphism. From [Ya1], it is known that if $x_8^2 \in \text{Im}(cl)$ for the cycle map cl , then we can show that cl itself is surjective. However it seems still unknown whether $x_8^2 \in \text{Im}(cl)$ or not.

Corollary 7.2. *Let $(G, p) = (F_4, 3)$. If $(**)$ in §5 is correct for some $n \geq 2$, then the cycle map $CH^*(BG) \rightarrow BP^*(BG) \otimes_{BP^*} \mathbb{Z}_{(3)}$ is surjective and*

$$CH^*(BG)/Tor \cong D \otimes (\mathbb{Z}_{(3)}\{1, 3x_4\} \oplus E).$$

Proof. The corollary follows from $|v_n x_8^2| = 16 - 2(3^n - 1) \leq 0$. \square

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